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The two isosceles triangles  $CJE$  and  $IJK$ , having equal angles at  $E$  and  $K$  respectively, are similar. Hence we have  $JE : CE :: JK : IK$ , or, using (1), (2) and (3)

$$h\sqrt{4 - h^2} : (3h - h^3) :: h : (2h - h^3).$$

Hence

$$(4') \quad (2 - h^2)\sqrt{4 - h^2} = 3 - h^2.$$

Squaring, we get

$$(4 - 4h^2 + h^4)(4 - h^2) = 9 - 6h^2 + h^4,$$

and expanding,

$$16 - 20h^2 + 8h^4 - h^6 = 9 - 6h^2 + h^4.$$

Finally, transposing and simplifying, we obtain the author's *Heptagon Cubic*:

$$(5) \quad 7 - 14h^2 + 7h^4 - h^6 = 0.$$

Solving by Horner's method, we find  $h^2 = .7530203962821\dots = \frac{3}{4}$ , approximately.

*Remark.* It will thus be seen that while there is introduced a new line  $IJ$ , we dispense with the consideration of the line  $OC$ , and with both the consideration and the computation of the author's lines  $SE$ ,  $SJ$ ,  $SK$  and  $SC$ . As a result, the equation (4') appears in a much simpler form than the author's equation (4).

The approximate construction of the heptagon may also be simplified as follows:

Let  $M$ ,  $N$  and  $P$  be three consecutive vertices of an inscribed regular hexagon. Draw the chord  $MP$  and the radius  $ON$ , and let  $MP$  meet  $ON$  in  $R$ . Then  $MR$  is, approximately, the length  $h$  of the side of the regular inscribed heptagon. The reason is self-evident: approximately,  $h = \frac{1}{2}\sqrt{3}$ , and  $MP$ , as the side of a regular inscribed triangle,  $= \sqrt{3}$ , so that  $MR = \frac{1}{2}\sqrt{3}$ , and therefore  $MR = h$ , approximately.

### A PROBLEM IN NUMBER THEORY.

By GEO. A. OSBORNE, Massachusetts Institute of Technology.

§ 1. When is the sum of the squares of two successive integers a perfect square? The following are examples:

$$3^2 + 4^2 = 5^2, \quad 20^2 + 21^2 = 29^2. \quad \text{The next is } 119^2 + 120^2 = 169^2.$$

The numbers 3, 20, 119, . . . are the terms of a series

$$0, 3, 20, 119, 696, \dots u_n, u_{n+1}, \tag{1}$$

where

$$u_{n+1} = 6u_n - u_{n-1} + 2. \tag{2}$$

This may be proved as follows:

From (2), which is the relation between any three successive terms of (1), we may derive the relation between any two successive terms as follows:

From (2)

$$\begin{aligned} u_{n+1} + u_{n-1} &= 6u_n + 2, \\ u_{n+1}^2 - u_{n-1}^2 &= (6u_n + 2)(u_{n+1} - u_{n-1}), \\ (u_{n+1} - 1)^2 - 6u_n u_{n+1} &= (u_{n-1} - 1)^2 - 6u_n u_{n-1}, \end{aligned}$$

Adding to each member  $(u_n - 1)^2$ , we have

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n u_{n+1} = (u_n - 1)^2 + (u_{n-1} - 1)^2 - 6u_{n-1} u_n, \quad (3)$$

which is of the form  $f(n) = f(n - 1)$ .

Hence by induction,  $f(n) = c$ , a constant independent of  $n$ . By applying the first member of (3) to the terms 3 and 20, we find  $c = 5$ . Hence

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n u_{n+1} = 5 \quad (4)$$

is the relation between any two successive terms of (1).

Solving (4) with respect to  $u_{n+1}$ , we have

$$u_{n+1} = 3u_n + 1 \pm \sqrt{2u_n^2 + 2u_n + 1}, \quad (5)$$

in which the lower sign is rejected since, otherwise, the right member would be less than  $u_n$ . It follows from (5) that

$$2u_n^2 + 2u_n + 1 = \text{a square},$$

that is,

$$u_n^2 + (u_n + 1)^2 = \text{a square},$$

one part of the result which was to be proved.

§ 2. From the terms of (1) we may write

$$\begin{aligned} 0^2 + 1^2 &= 1^2, \\ 3^2 + 4^2 &= 5^2, \\ 20^2 + 21^2 &= 29^2, \\ 119^2 + 120^2 &= 169^2, \\ 696^2 + 697^2 &= 985^2, \\ 4059^2 + 4060^2 &= 5741^2, \\ 23660^2 + 23661^2 &= 33461^2, \\ 137903^2 + 137904^2 &= 195025^2, \\ 803760^2 + 803761^2 &= 1136689^2, \\ \cdot &\quad \cdot &\quad \cdot &\quad \cdot &\quad \cdot &\quad \cdot &\quad \cdot \end{aligned}$$

The second members are terms of a series,

$$1, 5, 29, \dots u_n, u_{n+1},$$

where

$$u_{n+1} = 6u_n - u_{n-1}.$$

§ 3. It remains to be shown that the terms of the series (1) are the *only* integers that satisfy the condition

$$N^2 + (N+1)^2 = \text{a square.} \quad (6)$$

If from (4) we express  $u_n$  in terms of  $u_{n+1}$ , we have

$$u_n = 3u_{n+1} + 1 - 2\sqrt{2u_{n+1}^2 + 2u_{n+1} + 1}, \quad (7)$$

from which

$$u_{n-1} = 3u_n + 1 - 2\sqrt{2u_n^2 + 2u_n + 1}. \quad (8)$$

Consider the equation

$$y = 3x + 1 - 2\sqrt{2x^2 + 2x + 1}. \quad (9)$$

Then

$$\begin{aligned} 2y^2 + 2y + 1 &= 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1})^2 + 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1}) + 1 \\ &= (4x + 2 - 3\sqrt{2x^2 + 2x + 1})^2. \end{aligned}$$

Hence if

$$2x^2 + 2x + 1 = \text{a square,}$$

then also

$$2y^2 + 2y + 1 = \text{a square.}$$

That is, if  $x$  satisfies (6), so does  $y$ . Comparing (9) with (7) and (8), it appears that if  $x = u_{n+1}$ ,  $y = u_n$ ; and if  $x = u_n$ ,  $y = u_{n-1}$ . And as  $y$  is an increasing function of  $x$ , since

$$\frac{dy}{dx} = 3 - \frac{4x+2}{\sqrt{2x^2 + 2x + 1}} > 0,$$

it follows that if  $u_n < x < u_{n+1}$ , then  $u_{n-1} < y < u_n$ . That is, if there is an integer satisfying (6) between  $u_n$  and  $u_{n+1}$ , there is another such integer between  $u_{n-1}$  and  $u_n$ .

There is no integer satisfying (6) between the terms 3 and 20; hence there is none between 20 and 119, and consequently none between any two successive terms of the series (1).

§ 4. In the list of equations in § 2, it may be noticed that none of the successive integers end in 2, 5 or 8. As the final digits recur, it follows that

*The sum of the squares of two successive integers, one of which ends in 2, 5 or 8, cannot be a perfect square.*